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The uniform motion of rectangular and parabolic punches in a viscoelastic layer☆

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ABSTRACT

The plane contact problem of the interaction of a rigid moving punch with a viscoelastic layer is considered. It is assumed that the punch moves at a constant speed and is forced into the layer by a constant normal force. There is no friction in the contact area between the stamp and the surface of the layer. The displacement of the boundary of the layer, due to the normal load applied to it, is determined. The integral equation of the contact problem for determining the contact pressure is then derived and an approximate solution of this integral equation is constructed using the Multhopp–Kalandiya method for a rectangular punch and one of the modifications of the method of orthogonal polynomials for a parabolic punch.

Contact problems for viscoelastic bodies have been treated in different formulations (see Refs 1–4, etc.). The contact problem of the motion of rigid punches in the boundary of a viscoelastic layer is investigated below for the first time.

1. Solution of the auxiliary problem

Suppose a punch moves at a constant speed in the boundary of a viscoelastic layer of thickness h. We shall consider the motion of the load in a system of coordinates which moves at a constant velocity $\mathbf{V}$ in the negative direction of the X axis. In this case, the displacement and load $q(x)$ are independent of time. We will first derive the relation between the displacement and the pressure on the upper boundary of the elastic layer (Fig. 1).

We write the Lamé equations and the boundary conditions for the plane stress-strain state

$$2(1-\nu)\operatorname{grad}\operatorname{div}\mathbf{U}-(1-2\nu)\operatorname{rot}\operatorname{rot}\mathbf{U}=0 \tag{1.1}$$

$$u(x,0)=\upsilon(x,0)=0,\quad \tau_{xy}(x,h)=0,\quad |x|<\infty$$
$$\sigma_y(x,h)=-q(x),\quad |x|\le a;\quad \sigma_y(x,h)=0,\quad |x|>a \tag{1.2}$$

Here u and υ are the components of the displacement vector $\mathbf{U}$ along the X and Y axes, and σ_y and τ_{xy} are the normal and shear stresses.

Solving problem (1.1), (1.2) using a Fourier transformation with respect to x, we obtain the following expression for the vertical displacements

$$\upsilon(x,h)=-\frac{1-\nu}{\pi G}\int_{-a}^{a}q(\xi)K\left(\frac{\xi-x}{h}\right)d\xi$$

$$K(t)=\int_0^\infty\frac{L(u)}{u}\cos ut\,du,\quad L(u)=\frac{2\kappa\operatorname{sh}2u-4u}{2\kappa\operatorname{ch}2u+\kappa^2+1+4u^2},\quad \kappa=3-4\nu \tag{1.3}$$

where G and ν are the shear modulus and Poisson's ratio.

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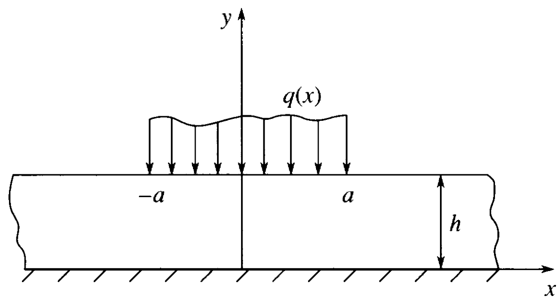

Fig. 1.

In the elastic formulation of the problem, the expression for the vertical displacements has the form (1.3), and, in the case of a viscoelastic layer, instead of expression (1.3), by analogy with what has been indicated earlier (Ref. 1, formula 49.2)) we have

$$\upsilon(x,h) = -\frac{1}{\pi\Theta_f}\int_{-a}^{a} q(\xi)\left[K\left(\frac{\xi-x}{h}\right)+\int_{-\infty}^{0} l(-\tau)K\left(\frac{\xi-x-V\tau}{h}\right)d\tau\right]d\xi$$

$$\Theta_f = \frac{G_f}{1-\nu},\quad l(t) = k\exp\left(-\frac{t}{\lambda}\right),\quad k = \frac{1}{\beta}\left(1-\frac{\beta}{\lambda}\right) \tag{1.4}$$

Here, λ and β are constants and G_f is the instantaneous shear modulus.

2. Derivation of the integral equation

We now consider to the contact problem. In this problem, the vertical displacement $v(x, h)$ is known and is equal to the indentation of the punch $-\delta(x)$ under the action of a force P, while the contact pressure $q(x)$ is unknown.

We now introduce the dimensionless quantities and notation

$$\varphi(x') = \frac{q(x)}{\Theta_f},\quad \varepsilon = \frac{h}{a},\quad \mu = \frac{\lambda V}{a},\quad g(x') = \frac{\delta(x)}{a}$$

$$x' = \frac{x}{a},\quad \xi' = \frac{\xi}{a},\quad \tau' = \frac{\tau}{\lambda},\quad w' = \xi' - x'$$

We write Eq. (1.4) in the form (the primes are omitted)

$$\int_{-1}^{1}\varphi(\xi)\left[K_1\left(\frac{w}{\varepsilon}\right)+k\lambda K_2\left(\frac{w}{\varepsilon}\right)\right]d\xi = \pi g(x) \tag{2.1}$$

where

$$K_1\left(\frac{w}{\varepsilon}\right) = -\ln\left|\frac{w}{\varepsilon}\right| + F_1\left(\frac{w}{\varepsilon}\right),\quad F_1\left(\frac{w}{\varepsilon}\right) = -\int_0^\infty \frac{[1-L(u)]\cos\frac{uw}{\varepsilon}-e^{-u}}{u}du \tag{2.2}$$

$$K_2\left(\frac{w}{\varepsilon}\right) = \varepsilon^2\int_0^\infty \frac{L(u)}{u(\varepsilon^2+\mu^2u^2)}\cos\frac{uw}{\varepsilon}du - \varepsilon\mu\int_0^\infty \frac{L(u)}{\varepsilon^2+\mu^2u^2}\sin\frac{uw}{\varepsilon}du \tag{2.3}$$

In expression (2.2), the logarithmic term has been separated out using the representation

$$\ln|t| = \int_0^\infty \frac{e^{-u}-\cos ut}{u}dt$$

Since the function $1-L(u)$ behaves as e^{-2u} at infinity, it is advisable to separate out $1-L(u)$ in order to improve the convergence of the integrals.

After some reduction, instead of Eq. (2.1) we have

$$\int_{-1}^{1}\varphi(\xi)K(w)d\xi = \pi g(x), \quad K(w) = -\ln\left|\frac{w}{\varepsilon}\right| + F(w)$$

$$F(w) = F_1\left(\frac{w}{\varepsilon}\right) + k\lambda\left[\sum_{i=1}^{3}F_i\left(\frac{w}{\varepsilon}\right) + F_4(w) - \ln\frac{\mu}{\varepsilon}\right] \tag{2.4}$$

where

$$F_2\left(\frac{w}{\varepsilon}\right) = \mu^2\int_0^{\infty}\frac{1-L(u)}{\varepsilon^2+\mu^2u^2}u\cos\frac{uw}{\varepsilon}du, \quad F_3\left(\frac{w}{\varepsilon}\right) = \varepsilon\mu\int_0^{\infty}\frac{1-L(u)}{\varepsilon^2+\mu^2u^2}\sin\frac{uw}{\varepsilon}du$$

$$F_4\left(\frac{w}{\mu}\right) = \exp\left(\frac{w}{\mu}\right)\mathrm{Ei}\left(-\frac{w}{\mu}\right) - \ln\left|\frac{w}{\mu}\right| \tag{2.5}$$

Here EI(x) is the integral exponential function.[5]

3. The solution of the contact problem for a rectangular punch

In the case of a rectangular punch with sharp edges, the solution of the integral Eq. (2.4) can be represented in the form (Ref. 6, Theorem 2.1)

$$\varphi(x) = \Phi(x)(1-x^2)^{-1/2}, \quad \Phi(x)\in C_n(-1,1) \tag{3.1}$$

For the problem in question $g(x)\equiv g$. We shall solve Eq. (2.4) using the modified Multhopp–Kalandiya method.[7] For $\Phi(x)$,we construct the Lagrange interpolation polynomial

$$\tilde{\Phi}(\vartheta) = \frac{1}{N}\sum_{n=1}^{N}\tilde{\Phi}(\theta_n)\left(1+2\sum_{m=1}^{N-1}\cos m\theta_n\cos m\vartheta\right), \quad \tilde{\Phi}(\vartheta)\equiv\Phi(\cos(\vartheta)) \tag{3.2}$$

using the interpolation points

$$x_n = \cos\theta_n, \quad \theta_n = \pi(n-1/2)/N; \quad n = 1, 2, \ldots, N \tag{3.3}$$

which are the zeroes of the Chebyshev polynomial $T_N(x)$.

Changing to new variables using the formulae $x=\cos\vartheta$, $\xi=\cos\psi$, we write Eq. (2.4) in the form

$$\int_0^{\pi}\tilde{\Phi}(\psi)\ln\left|\frac{\cos\psi-\cos\vartheta}{\varepsilon}\right|d\psi = \pi g - \int_0^{\pi}\tilde{\Phi}(\psi)F(\cos\psi-\cos\vartheta)d\psi, \quad \vartheta\in[0,\pi] \tag{3.4}$$

We now substitute expression (3.2) into the last equation and, using the relation

$$-\int_0^{\pi}\cos s\psi\ln\left|\frac{\cos\psi-\cos\vartheta}{\varepsilon}\right|d\psi = \begin{cases}\pi\ln 2\varepsilon, & s = 0\\ \pi s^{-1}\cos s\vartheta, & s\neq 0\end{cases} \tag{3.5}$$

and Gauss's quadrature formula

$$\int_0^{\pi}\chi(\psi)d\psi = \frac{\pi}{N}\sum_{n=1}^{N}\chi(\vartheta_n) \tag{3.6}$$

we rewrite Eq. (3.4) in the form

$$\sum_{n=1}^{N}\tilde{\Phi}(\theta_n)\left[\ln(2\varepsilon)+2\sum_{m=1}^{N-1}\frac{\cos m\theta_n\cos m\vartheta}{m}\right] = Ng - \sum_{n=1}^{N}\tilde{\Phi}(\theta_n)F(\cos\theta_n-\cos\vartheta)$$

We use the collocation method, that is, assuming $\vartheta=\theta_k$, we obtain a system of linear algebraic equations for determining of $\tilde{\Phi}(\theta_n)$

$$\sum_{n=1}^{N}\tilde{\Phi}(\theta_n)\left[\ln(2\varepsilon)+2\sum_{m=1}^{N-1}\frac{\cos m\theta_n\cos m\theta_k}{m}+F(\cos\theta_n-\cos\theta_k)\right] = Ng$$

$$k = 1, 2, \ldots, N \tag{3.7}$$

4. The solution of the contact problem for a parabolic punch with smooth edges

We rewrite Eq. (2.4) in the form

$$-\int_{-1}^{1}\varphi(\xi)\ln\left|\frac{\xi-x}{\varepsilon}\right|d\xi = \pi f(x);\quad f(x) = g(x)-\frac{1}{\pi}\int_{-1}^{1}\varphi(\xi)F(\xi-x)d\xi \tag{4.1}$$

It is assumed that the system of coordinates is arranged such that the contact area is a symmetric segment and that the function $g(x)$ in this system of coordinates has the form

$$g(x) = g_0+\gamma x-x^2/(2\rho) \tag{4.2}$$

Here g_0 is the indentation of the punch, γ is the angle of rotation and ρ is the radius of curvature at the vertex of the parabolic punch divided by the half length of the contact area a. Since the edges of the punch are smooth, the pressure on the boundary of the contact area is equal to zero.

We now transform the singular operator in Eq. (4.1) subject to the conditions

$$\varphi(\pm 1) = 0 \tag{4.3}$$

We obtain the equation[6]

$$\varphi(x) = -\frac{1}{\pi}\sqrt{1-x^2}\int_{-1}^{1}\frac{f'(\xi)}{(\xi-x)\sqrt{1-\xi^2}}d\xi;\quad f'(\xi) = g'(\xi)-\frac{1}{\pi}\int_{-1}^{1}\varphi(\tau)\frac{dF(\tau-\xi)}{d\xi}d\tau \tag{4.4}$$

Here,

$$\frac{dF(\tau-\xi)}{d\xi} = -\frac{dF(w)}{dw} = -\frac{k\lambda}{\mu}\ln\left|\frac{w}{\mu}\right|+G(w)$$

$$G(w) = -F_1'\left(\frac{w}{\varepsilon}\right)-k\lambda\left\{\sum_{i=1}^{3}F_i'\left(\frac{w}{\varepsilon}\right)+\frac{1}{\mu}F_4\left(\frac{w}{\mu}\right)\right\},\quad G(w)\in C(-2,2)$$

Eq. (4.4) is equivalent to (4.1) subject to two additional conditions, written in dimensionless form, which replace (4.3),

$$\frac{P}{\Theta_f a}+\int_{-1}^{1}\frac{f'(\xi)\xi}{\sqrt{1-\xi^2}}d\xi = 0,\quad \int_{-1}^{1}\frac{f'(\xi)}{\sqrt{1-\xi^2}}d\xi = 0 \tag{4.5}$$

The force P is obviously connected with the contact pressure $q(x)$ by the relation

$$P = \int_{-a}^{a}q(\xi)d\xi \tag{4.6}$$

To these conditions, it is necessary to add the relation between the force P and the indentation g_0

$$\frac{P}{\Theta_f a} = \frac{1}{\ln 2\varepsilon}\int_{-1}^{1}\frac{f(\xi)}{\sqrt{1-\xi^2}}d\xi \tag{4.7}$$

and the condition for the moments

$$\frac{Pe}{\Theta_f a^2} = \int_{-1}^{1}f'(\xi)\sqrt{1-\xi^2}d\xi \tag{4.8}$$

from which the eccentricity e of the application of the force P can be determined.

It is well known that the function $\varphi(x)$ has the structure (Ref. 6, Theorem 2.5)

$$\varphi(x) = \omega(x)\sqrt{1-x^2},\quad \omega(x)\in C_n(-1,1) \tag{4.9}$$

We shall solve the equation in Fourier series, that is, in essence, we use a modified method of orthogonal polynomials.[6] Changing to the new variables

$$x = \cos\vartheta,\quad \xi = \cos\psi,\quad \tau = \cos\theta$$

we write Eq. (4.4) in the form

$$\varphi(\cos\vartheta) = -\frac{1}{\pi}\sin\vartheta\int_0^\pi \frac{f'(\cos\psi)}{\cos\psi-\cos\vartheta}d\psi$$

$$f'(\cos\psi) = g'(\cos\psi) - \frac{1}{\pi}\left[-\frac{k\lambda}{\mu}\int_0^\pi \varphi(\cos\theta)\ln\left|\frac{\cos\theta-\cos\psi}{\mu}\right|\sin\theta d\theta\right.$$

$$\left.+\int_0^\pi \varphi(\cos\theta)\tilde{G}(\theta,\psi)d\theta\right], \quad \tilde{F}(\theta,\psi) = G(\cos\theta-\cos\psi)\sin\theta \tag{4.10}$$

We now introduce the two orthonormal systems of functions

$$u_k(\vartheta) = \sqrt{\frac{2}{\pi}}\sin k\vartheta, \quad k = 1, 2, \ldots$$

$$\upsilon_0(\vartheta) = \frac{1}{\sqrt{\pi}}, \quad \upsilon_k(\vartheta) = \sqrt{\frac{2}{\pi}}\cos k\vartheta, \quad k = 1, 2, \ldots \tag{4.11}$$

and expand the functions $\varphi(\cos\vartheta)$, $g'(\cos\vartheta)$, $\tilde{G}(\theta,\psi)$ in series. We obtain

$$\varphi(\cos\vartheta) = \sum_{n=1}^{\infty}\varphi_n u_n(\vartheta),\; g'(\cos\vartheta) = \sum_{n=0}^{\infty} g_n\upsilon_n(\vartheta),\; \tilde{G}(\theta,\psi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} G_{mn}\upsilon_m(\psi)u_n(\theta) \tag{4.12}$$

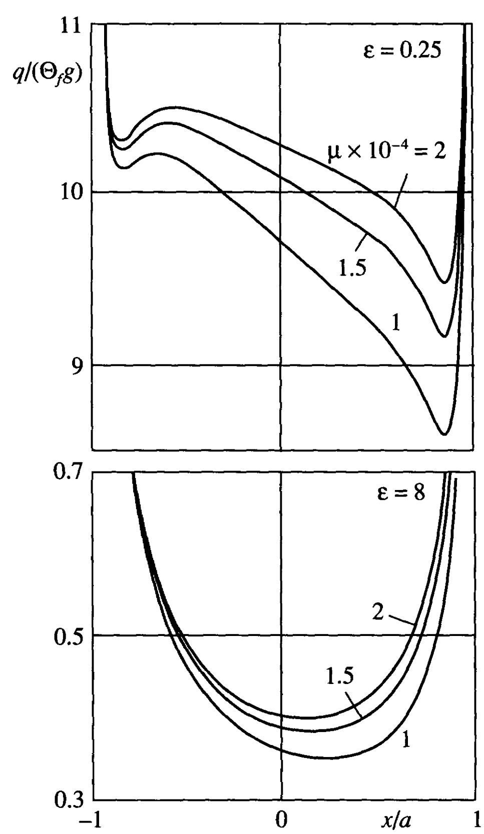

Fig. 2.

Substituting series (4.12) into the second formula of (4.10) and using relation (3.5) and the orthogonality property of the function (4.11), after reduction we have

$$f'(\cos\psi) = \sum_{m=0}^{\infty} g_m \upsilon_m(\vartheta) - \frac{k\lambda}{\mu}\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty}\varphi_n a_{mn}\right)\upsilon_m(\psi) - \frac{1}{\pi}\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty}\varphi_n G_{mn}\right)\upsilon_m(\psi) \quad (4.13)$$

The matrix a_{mn} has the following non-zero elements

$$a_{01} = \frac{\ln 2\mu}{\sqrt{2}}, \quad a_{12} = \frac{1}{2}, \quad a_{m\,m-1} = -\frac{1}{2m}, \quad a_{m\,m+1} = \frac{1}{2m}, \quad m = 2, 3, \ldots$$

We substitute expression (4.13) and the expansion for $\varphi(\cos\vartheta)$ in the first formula of (4.10) and, using the relation

$$\int_0^{\pi}\frac{\cos s\psi}{\cos\psi - \cos\vartheta}d\psi = \pi\frac{\sin s\vartheta}{\sin\vartheta}$$

we obtain

$$\sum_{m=1}^{\infty}\left[\varphi_m - \sum_{n=1}^{\infty}\left(\frac{k\lambda}{\mu}a_{mn} + \frac{1}{\pi}G_{mn}\right)\varphi_n\right]u_m(\vartheta) = \sum_{m=1}^{\infty} g_m u_m(\vartheta) \quad (4.14)$$

Equating the coefficients of like functions, we obtain a system of linear algebraic equations for determining of φ_n.

Determining φ_n from the system of linear algebraic equations, we find the required solution of the contact problem using formula (4.12). Using formula (4.13), we then determine the form of the function $f'(\cos\psi)$ and, afterwards, we find the approximate solution for the half length of the contact segment a, the angle or rotation of the punch γ and the eccentricity e of the application of the force P.

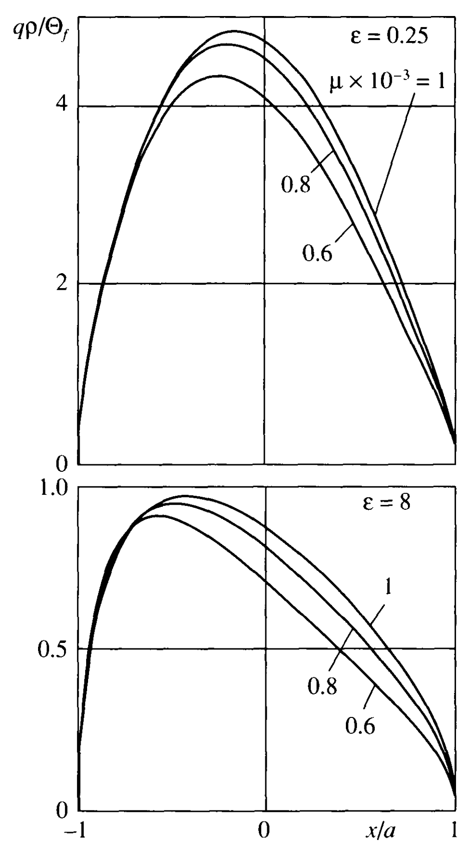

Fig. 3.

In order to find the relation between the force P and the indentation g_0, it is necessary to substitute the function $\varphi(\cos\vartheta)$ into the second formula of (4.1), having first changed the variables and expanded $F(\xi - x)$ using the function (4.11):

$$F(\cos\theta - \cos\psi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} F_{mn}\upsilon_m(\psi)u_n(\theta)$$

and then substitute it into condition (4.7).

5. Examples

We will now apply the relations obtained in the case when $\lambda/\beta = 1001$ and $\kappa = 1.8$, which corresponds to one type of rubber.

The contact pressure distributions for a rectangular punch are shown in Fig. 2 and for a parabolic punch in Fig. 3, for values of the relative thickness of the layer $\varepsilon = 0.25$ and $\varepsilon = 8$ and different values of the dimensionless parameter μ.

The values of the calculated quantities for a parabolic punch are presented below

	when $\varepsilon = 0.25$			when $\varepsilon = 8$		
$\mu \times 10^{-2}$	6	8	10	6	8	10
$a\sqrt{\frac{\Theta_f}{PR}} \times 10^2$	41.31	39.50	38.85	90.94	86.34	83.98
$\gamma\sqrt{\frac{\Theta_f R}{P}} \times 10^2$	19.93	15.66	12.74	131.27	103.80	85.42
$e\sqrt{\frac{\Theta_f}{PR}} \times 10^2$	−4.34	−3.32	−2.66	−16.89	−12.56	−9.98
$g\sqrt{\frac{\Theta_f R}{P}} \times 10^2$	40.95	35.82	32.43	424.45	355.49	310.37

The pressure distribution on the boundary of the layer when its relative thickness is increased tends to the pressure distribution on the boundary of the half-space.

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